

Renormalisation of supersymmetric gauge theory in the uneliminated component formalism

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We show that the renormalisation of the $\mathcal{N} = 1$ supersymmetric gauge theory when working in the component formalism, without eliminating auxiliary fields and using a standard covariant gauge, requires a non-linear renormalisation of the auxiliary fields.

1. Introduction

The renormalisation of $\mathcal{N} = 1$ supersymmetric gauge theory is certainly well-understood in the superfield formalism both in terms of formal analysis (for example Ref. [1]) and practical calculations (for example Ref. [2]). In accordance with the non-renormalisation theorem the superpotential is unrenormalised, leading to the standard expression for the Yukawa-coupling β -function in terms of the chiral superfield anomalous dimension. However, a feature of the superfield formalism which is often overlooked is the necessity for a non-linear renormalisation of the vector superfield [3].

In fact, as we shall see, the renormalisation program is perhaps most straightforwardly implemented in terms of component fields and in the case where the auxiliary fields F and D are eliminated using their equations of motion. It is well-documented in this case that the Lagrangian is multiplicatively renormalisable. From a practical point of view, moreover, although a softly-broken supersymmetric theory *can* be treated using superfields via spurion techniques, calculations in the MSSM are generally carried out using the eliminated component formalism. Now since the elimination of F and D gives rise to non-linear terms in the supersymmetry transformations of the physical fields, one might expect that the renormalisation program would be at least as simple in terms of the uneliminated formalism. Indeed, the uneliminated formalism has been employed for effective potential calculations [4] and in calculations of the β -function for soft (mass)² terms [5]. Our purpose here is simply to show how the uneliminated formalism requires some care in that (in a conventional covariant gauge) the theory is once again not multiplicatively renormalisable in the conventional sense; additional counter-terms are required which do not correspond to terms in the original Lagrangian but which can be generated by non-linear field renormalisations. However, these non-linear field renormalisations appear to be distinct from those of Ref. [3], since they only appear in the presence of chiral matter whereas the latter arise even in the pure gauge case.

We also consider what happens in the light-cone gauge, which is, in a sense we shall explain, “more supersymmetric” than the conventional covariant gauge [6].

2. Renormalisation

The Lagrangian is given in components by

$$\begin{aligned}
S_{\text{unel}} = \int d^4x & \left[-\frac{1}{4} F^{\mu\nu A} F_{\mu\nu}^A - i \bar{\lambda}^A \bar{\sigma}^\mu (D_\mu \lambda^A) + \frac{1}{2} (D^A)^2 \right. \\
& + F^i F_i - i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi - D^\mu \bar{\phi} D_\mu \phi + g \bar{\phi} R^A \phi D^A + i \sqrt{2} g (\bar{\phi} \lambda \psi - \bar{\psi} \bar{\lambda} \phi) \\
& \left. + F^i W_i + F_i W^i - \frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} W^{ij} \psi_i \psi_j \right],
\end{aligned} \tag{2.1}$$

where

$$W(\phi) = \frac{1}{6} Y^{ijk} \phi_i \phi_j \phi_k \quad (2.2)$$

is the superpotential, assumed cubic in ϕ for renormalisability, $W_i \equiv \frac{\partial W}{\partial \phi^i}$, and the lowering of indices indicates complex conjugation, so that $W^i = (W_i)^*$. For simplicity we omit possible linear and quadratic terms. The chiral fields transform according to a representation R of the gauge group and we write $\lambda = \lambda^A R^A$. If we eliminate the auxiliary fields F and D using their equations of motion:

$$\begin{aligned} D^A + g \bar{\phi} R^A \phi &= 0, \\ F^i + W^i &= 0, \end{aligned} \quad (2.3)$$

we obtain the eliminated Lagrangian, given in components by

$$\begin{aligned} S_{\text{el}} = \int d^4x & \left[-\frac{1}{4} F^{\mu\nu A} F_{\mu\nu}^A - i \bar{\lambda}^A \bar{\sigma}^\mu (D_\mu \lambda^A) \right. \\ & - i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi - D^\mu \bar{\phi} D_\mu \phi - \frac{1}{2} g^2 (\bar{\phi} R^A \phi) (\bar{\phi} R^A \phi) + i \sqrt{2} g (\bar{\phi} \lambda \psi - \bar{\psi} \bar{\lambda} \phi) \\ & \left. - W^i W_i - \frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} W^{ij} \psi_i \psi_j \right]. \end{aligned} \quad (2.4)$$

In either case we use the standard gauge-fixing term

$$S_{\text{gf}} = \frac{1}{2\alpha} \int d^4x (\partial \cdot A)^2 \quad (2.5)$$

with its associated ghost terms. The theory in the eliminated case is rendered finite by replacing fields and couplings in Eq. (2.4) by their corresponding bare versions. We have

$$\begin{aligned} \lambda_B^A &= Z_\lambda^{\frac{1}{2}} \lambda^A, \quad A_{\mu B}^A = Z_A^{\frac{1}{2}} A_\mu^A, \quad \phi_B = Z_\phi^{\frac{1}{2}} \phi, \quad \psi_B = Z_\psi^{\frac{1}{2}} \psi, \\ g_B &= Z_g g, \quad Y_B^{ijk} = \left(Z_\Phi^{-\frac{1}{2}} \right)_l^i \left(Z_\Phi^{-\frac{1}{2}} \right)_m^j \left(Z_\Phi^{-\frac{1}{2}} \right)_n^k Y^{lmn}. \end{aligned} \quad (2.6)$$

Here Z_Φ is the renormalisation constant for the chiral superfield Φ so that the result for Y_B is the consequence of the non-renormalisation theorem. In general, however, when working in a standard covariant gauge in components, $Z_{\phi, \psi, \Phi}$ are all different; at one loop, in fact, we have

$$\begin{aligned} Z_\lambda &= 1 - 2g^2 L [\alpha C(G) + T(R)], \\ Z_A &= 1 + g^2 L [(3 - \alpha) C(G) - 2T(R)], \\ Z_g &= 1 + g^2 L [T(R) - 3C(G)], \\ Z_\phi &= 1 + L [-Y^2 + 2(1 - \alpha) g^2 C(R)], \\ Z_\psi &= 1 + L [-Y^2 - 2(1 + \alpha) g^2 C(R)], \\ Z_\Phi &= 1 + L [-Y^2 + 4g^2 C(R)], \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} (Y^2)^i_j &= Y^{ikl} Y_{jkl} \\ C(R) &= R^A R^A, \quad T(R) \delta^{AB} = \text{Tr}[R^A R^B], \end{aligned} \quad (2.8)$$

$C(G)$ is the adjoint Casimir and (using dimensional regularisation with $d = 4 - \epsilon$) $L = \frac{1}{16\pi^2\epsilon}$. But now what happens if we work with the uneliminated form of the action? We might expect the theory to be rendered finite by replacing fields and couplings in Eq. (2.1) by corresponding bare versions (now we also need $F_B = (Z_F)^{\frac{1}{2}} F$, $D_B = (Z_D)^{\frac{1}{2}} D$ of course). It is not difficult to see, however, that there are one-loop diagrams with 2 ϕ and 2 $\bar{\phi}$ external fields for which there are no counter-term diagrams in this case (while in the eliminated case, counterterms are supplied by the $W^i W_i$ term). We also find that the $F\phi^2$ and $D\bar{\phi}\phi$ terms are not rendered finite by the renormalisation constants given above.



Fig. 1: Diagrams with one F , two scalar lines. Dashed, full, double full, wavy, full/wavy, zigzag lines represent ϕ , ψ , F , A , λ , D propagators respectively.

To be precise, the results for the graphs in Fig. 1 are:

$$\Gamma_a^1 = -\frac{1}{2}\alpha L g^2 (Y_{ijk}[C(R)F]^i \phi^j \phi^k - 2Y_{ijk}F^i[C(R)\phi]^j \phi^k), \quad (2.9a)$$

$$\Gamma_b^1 = -\frac{1}{2}L g^2 (Y_{ijk}[C(R)F]^i \phi^j \phi^k - 2Y_{ijk}F^i[C(R)\phi]^j \phi^k), \quad (2.9b)$$

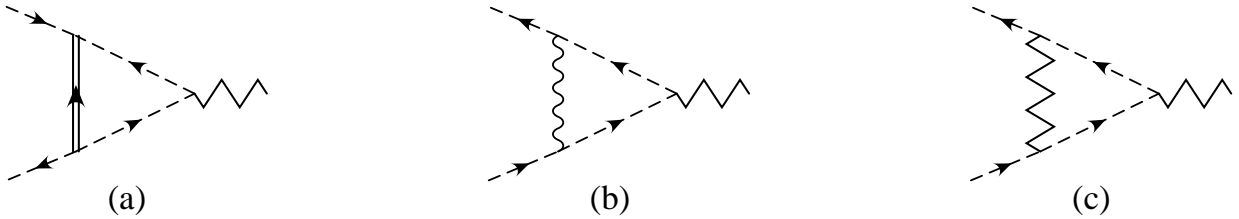


Fig. 2: Diagrams with one D , two scalar lines.

and the results for the graphs in Fig. 2 are

$$\begin{aligned} \Gamma_a^2 &= L \bar{\phi} Y^2 D \phi, \\ \Gamma_b^2 &= 2\alpha L g^2 \bar{\phi} [C(R) - \frac{1}{2}C(G)] D \phi, \\ \Gamma_c^2 &= -2L g^2 \bar{\phi} [C(R) - \frac{1}{2}C(G)] D \phi, \end{aligned} \quad (2.10)$$

where $D = D^A R^A$.

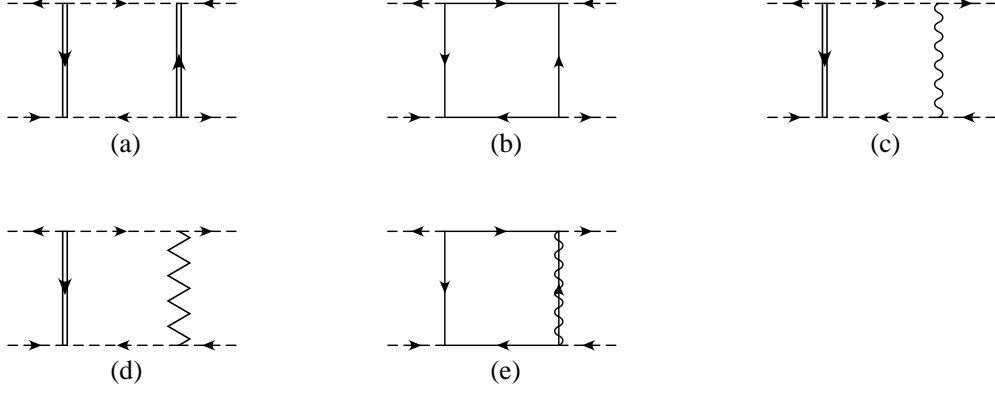


Fig. 3: Diagrams with 2 ϕ , 2 $\bar{\phi}$ lines and 2 or 4 Yukawa vertices.

The results for the graphs in Fig. 3 are

$$\begin{aligned}
\Gamma_a^3 &= LY_{imn}Y_{j pq}Y^{kmp}Y^{lnq}\phi^i\phi^j\phi_k\phi_l, \\
\Gamma_b^3 &= -LY_{imn}Y_{j pq}Y^{kmp}Y^{lnq}\phi^i\phi^j\phi_k\phi_l, \\
\Gamma_c^3 &= -\frac{1}{2}\alpha Lg^2Y_{ijm}Y^{kln}[C(R)]^m{}_n\phi^i\phi^j\phi_k\phi_l, \\
\Gamma_d^3 &= \frac{1}{2}Lg^2Y_{ijm}Y^{kln}[C(R)]^m{}_n\phi^i\phi^j\phi_k\phi_l, \\
\Gamma_e^3 &= -2Lg^2Y_{ijm}Y^{kln}[C(R)]^m{}_n\phi^i\phi^j\phi_k\phi_l.
\end{aligned} \tag{2.11}$$

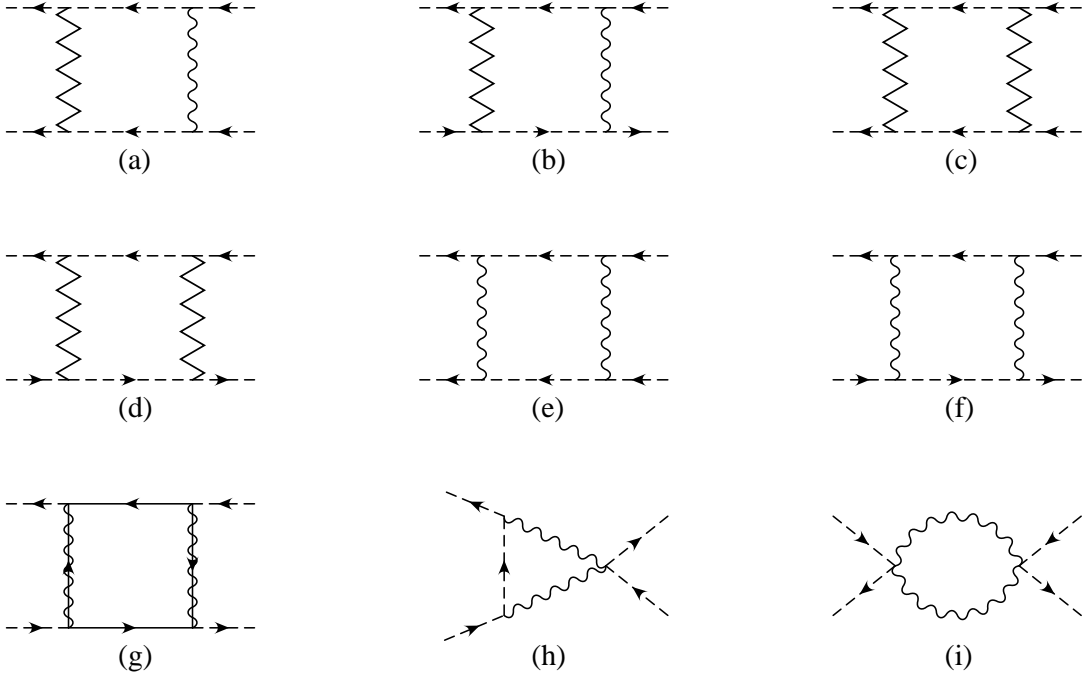


Fig. 4: Diagrams with 2 ϕ , 2 $\bar{\phi}$ lines and 4 gauge vertices.

The results for the graphs in Fig. 4 are

$$\begin{aligned}
\Gamma_a^4 &= 2\alpha g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^A R^B \phi), \\
\Gamma_b^4 &= -2\alpha g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^B R^A \phi), \\
\Gamma_c^4 &= g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^A R^B \phi), \\
\Gamma_d^4 &= g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^B R^A \phi), \\
\Gamma_e^4 &= \alpha^2 g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^A R^B \phi), \\
\Gamma_f^4 &= \alpha^2 g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^B R^A \phi), \\
\Gamma_g^4 &= -8g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^B R^A \phi), \\
\Gamma_h^4 &= -2\alpha^2 g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^A R^B \phi + \bar{\phi} R^B R^A \phi), \\
\Gamma_i^4 &= (3 + \alpha^2) g^4 L (\bar{\phi} R^A R^B \phi) (\bar{\phi} R^A R^B \phi + \bar{\phi} R^B R^A \phi).
\end{aligned} \tag{2.12}$$

The results for the graphs contributing to the remaining interaction terms in Eq. (2.1) are the same as in the eliminated case so we shall not give detailed results. The renormalisation constants $Z_{\phi, \psi, A}$ are also the same as in the eliminated case, and in addition we have

$$Z_F = 1 - LY^2, \quad Z_D = 1 - 2Lg^2 T(R). \tag{2.13}$$

We find that

$$\begin{aligned}
& \left[\Gamma_1 + \frac{1}{2} Y_{Bijk} F_B^i \phi_B^j \phi_B^k + (\text{c.c.}) \right] + \Gamma_2 + \Gamma_3 + \Gamma_4 + g_B \bar{\phi}_B D_B \phi_B \\
&= \left\{ \frac{1}{2} Y_{ijk} F^i \phi^j \phi^k - \frac{1}{2} (\alpha + 3) g^2 L \left[Y_{ijk} [C(R) F]^i \phi^j \phi^k \right. \right. \\
&+ \left. \frac{1}{2} Y_{ijm} Y^{klm} [C(R)]^m \phi^i \phi^j \phi_k \phi_l \right] + (\text{c.c.}) \left. \right\} \\
&+ g \bar{\phi} D \phi - (\alpha + 2) C(G) g^3 L [\bar{\phi} D \phi + g (\bar{\phi} R^A \phi) (\bar{\phi} R^A \phi)].
\end{aligned} \tag{2.14}$$

The residual divergence cancels if we substitute the equations of motion, Eq. (2.3), for D^A and F^i , as we would expect.

Alternatively, it is clear that these remaining divergences can all be cancelled by making the nonlinear renormalisations

$$\begin{aligned}
(F_B)_i &= (Z_F^{\frac{1}{2}} F)_i + \frac{1}{2} (\alpha + 3) g^2 L [C(R)]^l_i Y_{ljk} \phi^j \phi^k, \\
(D_B)^A &= (Z_D^{\frac{1}{2}} D)^A + (\alpha + 2) C(G) g^3 L \bar{\phi} R^A \phi.
\end{aligned} \tag{2.15}$$

A similar phenomenon was observed in a study of the renormalisation of $\mathcal{N} = \frac{1}{2}$ theories, presented in Refs. [7]; though in the case without a superpotential considered

there, application of the equation of motion for F is rather trivial, since the equation of motion for \bar{F} gives $F = 0$. In the $\mathcal{N} = \frac{1}{2}$ case, however, a further field redefinition (of the gaugino field λ) is necessary, and this redefinition has no analogy in the $\mathcal{N} = 1$ case considered here.

3. The light-cone gauge

It is interesting to reconsider the above calculations in the light-cone gauge, corresponding to the $\alpha \rightarrow 0$ limit of

$$S_{\text{gf}} = \frac{1}{2\alpha} \int d^4x (n \cdot A)^2, \quad \text{with } n^2 = 0. \quad (3.1)$$

In the light-cone gauge one again has a choice between an eliminated and an uneliminated formalism, distinct from that associated with the auxiliary fields of supersymmetry. Choosing $n = n^-$, the light-cone gauge corresponds to $A^+ = 0$ and the field A^- is non-propagating and can be eliminated by its equation of motion. Moreover, the condition $A^+ = 0$ is preserved by the subset of supersymmetry transformations corresponding to setting the infinitesimal spinor ϵ governing these transformations to be $\epsilon = \epsilon^+$. (This is reminiscent of $\mathcal{N} = \frac{1}{2}$ supersymmetry [8], where the action is invariant under supersymmetry transformations with respect to ϵ , but with $\bar{\epsilon} = 0$). As a consequence, one finds in the light-cone gauge that

$$\begin{aligned} Z_\lambda &= 1 - 2g^2 L [T(R) - 3C(G)], \\ Z_\phi &= Z_\psi = 1 + L [-Y^2 + 4g^2 C(R)], \end{aligned} \quad (3.2)$$

reflecting the preservation of (half the) supersymmetry by the gauge.

Light-cone gauge QCD was discussed in Ref. [9], where it was shown that a computation of the gauge two-point function in the A^- -uneliminated formalism leads to divergent structures not corresponding to terms in the Lagrangian, which however vanish if the equation of motion for A^- is applied. So this is completely analogous to the situation we found above.

Returning to the supersymmetric theory, we have recalculated Eq. (2.9) in the uneliminated light-cone gauge; Eq. (2.9b) is manifestly unchanged but $\Gamma_a^1 = -\Gamma_b^1$ so that there is no 1PI divergence, as in the superfield case. Z_ϕ now corresponds to the supersymmetric result (as indicated above in Eq. 3.2), but Z_F remains the same as in the covariant gauge case and so we obtain

$$\Gamma_1 + \frac{1}{2}Y_{Bijk}F_B^i\phi_B^j\phi_B^k + (\text{c.c.}) = \frac{1}{2}Y_{ijk}F^i\phi^j\phi^k - g^2LY_{ijk}[C(R)F]^i\phi^j\phi^k + (\text{c.c.}), \quad (3.3)$$

or more generally (instead of Eq. (2.14))

$$\begin{aligned} & \left[\Gamma_1 + \frac{1}{2}Y_{Bijk}F_B^i\phi_B^j\phi_B^k + (\text{c.c.}) \right] + \Gamma_2 + \Gamma_3 + \Gamma_4 + g_B\bar{\phi}_B D_B\phi_B \\ &= \left\{ \frac{1}{2}Y_{ijk}F^i\phi^j\phi^k - g^2L \left[Y_{ijk}[C(R)F]^i\phi^j\phi^k \right. \right. \\ & \quad \left. \left. + \frac{1}{2}Y_{ijm}Y^{kln}[C(R)]^m{}_n\phi^i\phi^j\phi_k\phi_l \right] + (\text{c.c.}) \right\} \\ & \quad + g\bar{\phi}D\phi - C(G)g^3L \left[\bar{\phi}D\phi + g \left(\bar{\phi}R^A\phi \right) \left(\bar{\phi}R^A\phi \right) \right]. \end{aligned} \quad (3.4)$$

Once again the residual divergence vanishes upon application of the equations of motion for F , D , or via a non-linear renormalisation corresponding to setting $\alpha = -1$ in Eq. (2.15).

4. Conclusions

We have seen that for $\mathcal{N} = 1$ theories the renormalisation program, when carried out in the F, D uneliminated formalism, contains some subtlety in that divergent terms of a form not present in the original Lagrangian are generated. These terms can, in fact, be eliminated either by means of non-linear field redefinitions (or renormalisations) or by imposing the equations of motion for F, D . We also recalled how an analogous phenomenon occurs in the light-cone gauge, where the rôle of the non-propagating F, D fields is played by the A^- gauge field. We believe that there is some pedagogical justification for clarifying these somewhat subtle features of the uneliminated form of the familiar $\mathcal{N} = 1$ supersymmetric theory. Moreover, this renders unsurprising the non-linear redefinition of \bar{F} found necessary in the $\mathcal{N} = \frac{1}{2}$ case[7]. In particular, it is interesting that in both cases the non-linear redefinition is gauge-parameter dependent. The phenomenon may also help to elucidate the additional redefinition of λ found to be required in the $\mathcal{N} = \frac{1}{2}$ case.

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